Treewidth Computation and Courcelle's Theorem in the CONGEST Model

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Algorithmic Meta Theorems

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- ✦ **We focus in graphs of bounded treewidth**

Algorithmic Meta Theorems

*Families of **properties, often defined by logical conditions**, can be **efficiently solved** on **certain classes of graphs.***

- **We focus in properties expressible in Monadic Second Order Logic**
- **We focus in graphs of bounded treewidth**
- $\tilde{O}(D)$ **rounds**

Monadic Second-Order Logic (MSO)

A property $\mathscr{P}$ is MSO-expressible if it can be defined using quantification over **vertices/edges** and over monadic **sets of vertices/edges.**

2-VERTEX COVER

Given a graph $G = (V, E)$ a vertex cover is a set $U \subseteq V$ such each each $uv \in E$ satisfies that $u \in S$ or $v \in S$

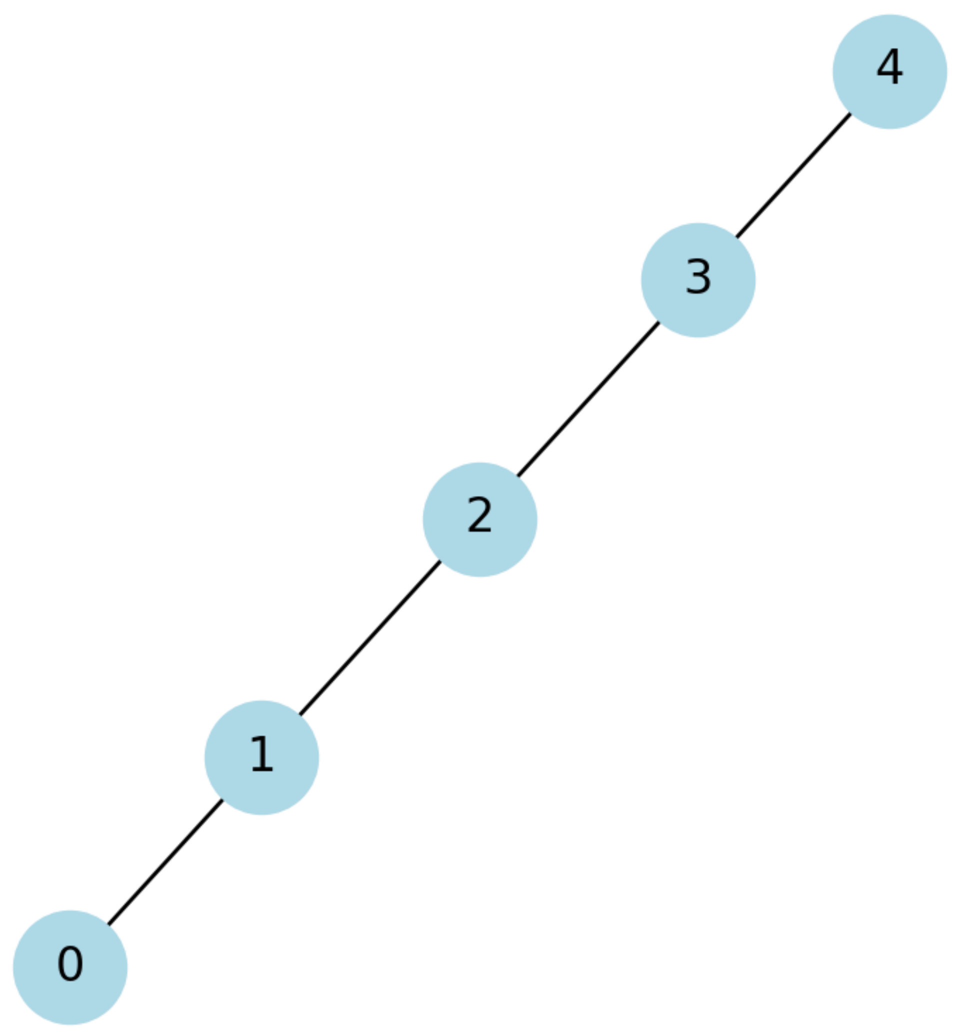
$$\text{(MSO)}\ \exists X \subseteq V, \exists x_1 \in V, \exists x_2 \in V, \forall y \in S, (y = x_1 \vee y = x_2)$$

$$\forall u \in V, \forall v \in V, (uv \in E \implies u \in X \vee v \in X)$$

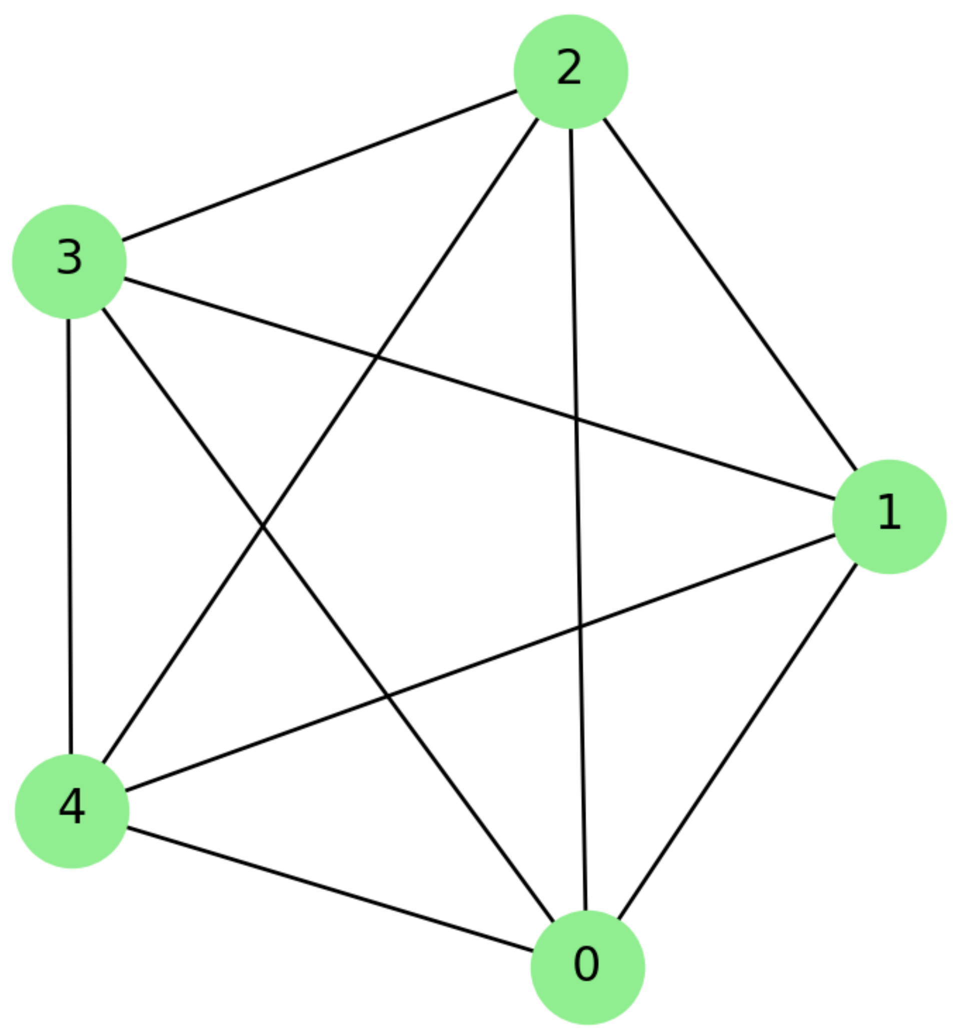
Optimization problem: Given G, find a vertex cover S of *minimum size*

Treewidth of a graph

- **Treewidth** is a **parameter** of a graph
- Characterizes how close a graph is to being a tree.



Treewidth 1



Treewidth $|V| - 1$

Class of k-bounded treewidth

$\downarrow$

Graphs with treewidth $\leq k$

Courcelle's Theorem

Courcelle's Theorem (Courcelle, 90'. Informal). Let φ be an MSO property and $k \in \mathbb{N}$. There exists an algorithm such that, for any graph G with $tw(G) \leq k$, decides if $G \models \varphi$ in linear time in the size of G.

Optimization version

Theorem (Borie, Parker and Tovey, 92'; Courcelle and Mosbah, 93'. Informal).

Given a graph optimization problem $\mathscr{P}$ of an MSO property and a class $\mathscr{G}$ of k-bounded treewidth graphs, there exists an algorithm such that, for any $G \in \mathscr{G}$, solves $\mathscr{P}(G)$ in linear time

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➡ **Optimization version of many NP problems solved in linear time in k-bounded treewidth graphs.**

Maximum Independent Set, Minimum Vertex Cover, etc...

➡ **Linear in $|G|$ but super-exponential dependance in k**

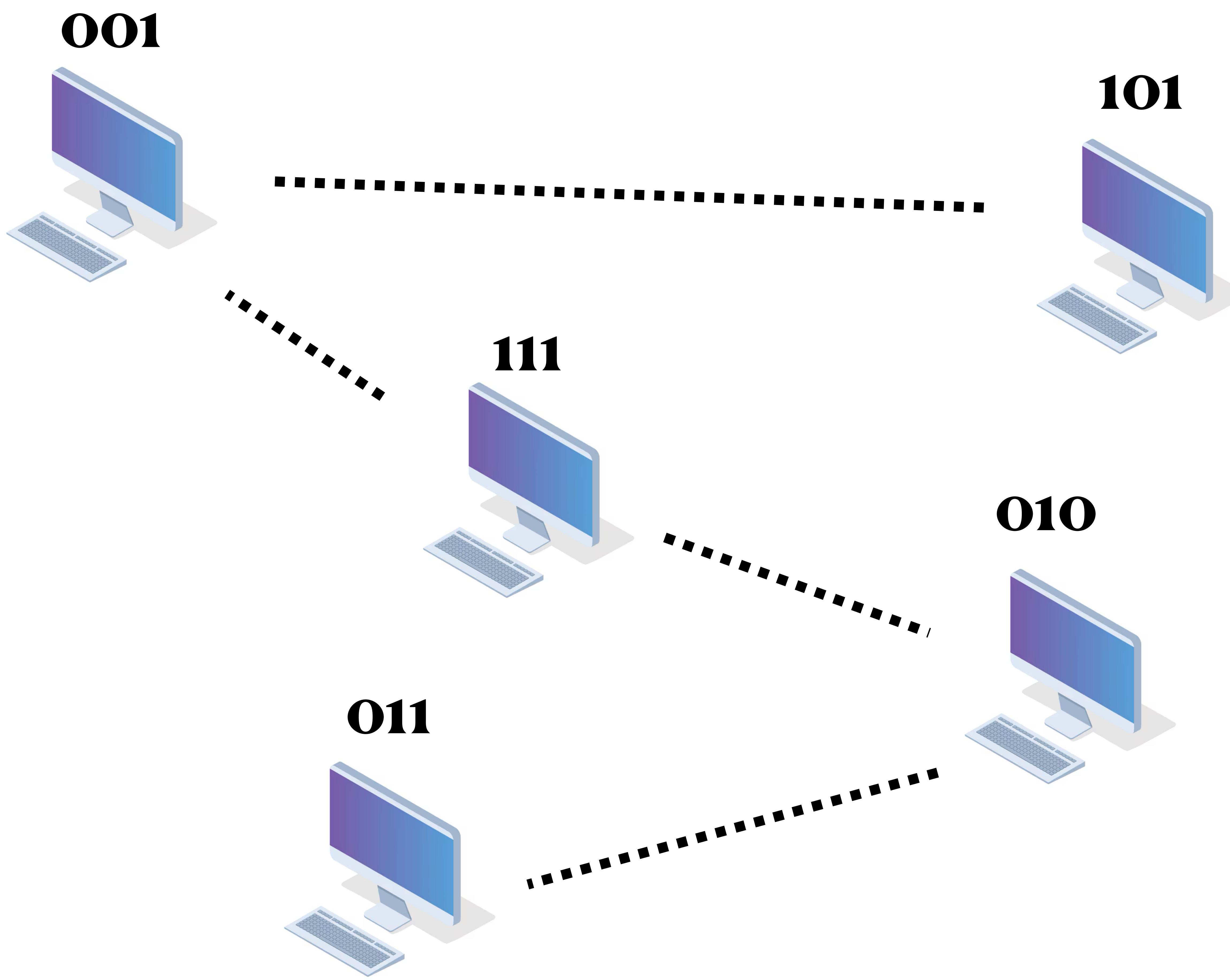
Distributed Computing

CONGEST Model

Distributed network represented as n-node graph $G = (V, E)$

- Each node has a unique **ID**.
- Nodes interact with their neighbors in **synchronous rounds.**
- Message size of $\mathcal{O}(\log n)$ bits.

- **Round complexity:** Maximum number of rounds a node in G sends a message to a neighbor.



Recent years

- Special interest in **sparse classes of graphs**
- Mainly **planar graphs**

Problem	General	Planar Graphs	Reference
Reachability	$\Omega(D+\sqrt{n}/\log n)$	$\tilde{O}(D)$	Parter, 2020.
MST	$\Omega(D+\sqrt{n})$	$\tilde{O}(D)$	Ghaffari and Haeupler, 2016.
Max-Flow	$\Omega(D+\sqrt{n})$	$\tilde{O}(D)$	Y. Abd-Elhaleem et al., 2025

MSO properties?

- Many of them are **hard to solve** in CONGEST.

Problem	General	Reference
Maximum IS	$\Omega(n^2/\log^2 n)$	Censor-Hillel, 2017.
Maximum Matching	$\tilde{\mathcal{O}}(\mu(G)^2)$	Ben-Basat et al et al. 2018
Minimum Vertex Cover	$\mathcal{O}(n^2/\log n)$	Folklore

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Can we improve them in bounded treewidth graph?

Our results

(Decision version)

Theorem (J. Li, P. Montealegre, I. Todinca, B.J.). Given a fixed integer k and an MSO-property φ, there is a deterministic CONGEST algorithm that, given an n-node graph with diameter D and treewidth at most k, decides $G \models \varphi$ in $\tilde{\mathcal{O}}(D)$ rounds.

(Optimization version)

Theorem (J. Li, P. Montealegre, I. Todinca, B.J.). Given a fixed integer k and an optimization problem $\mathcal{P}$ from an MSO-property φ, there is a deterministic CONGEST algorithm that, given an n-node graph with diameter D and treewidth at most k, computes $\mathcal{P}(G)$ in $\tilde{\mathcal{O}}(D)$ rounds.

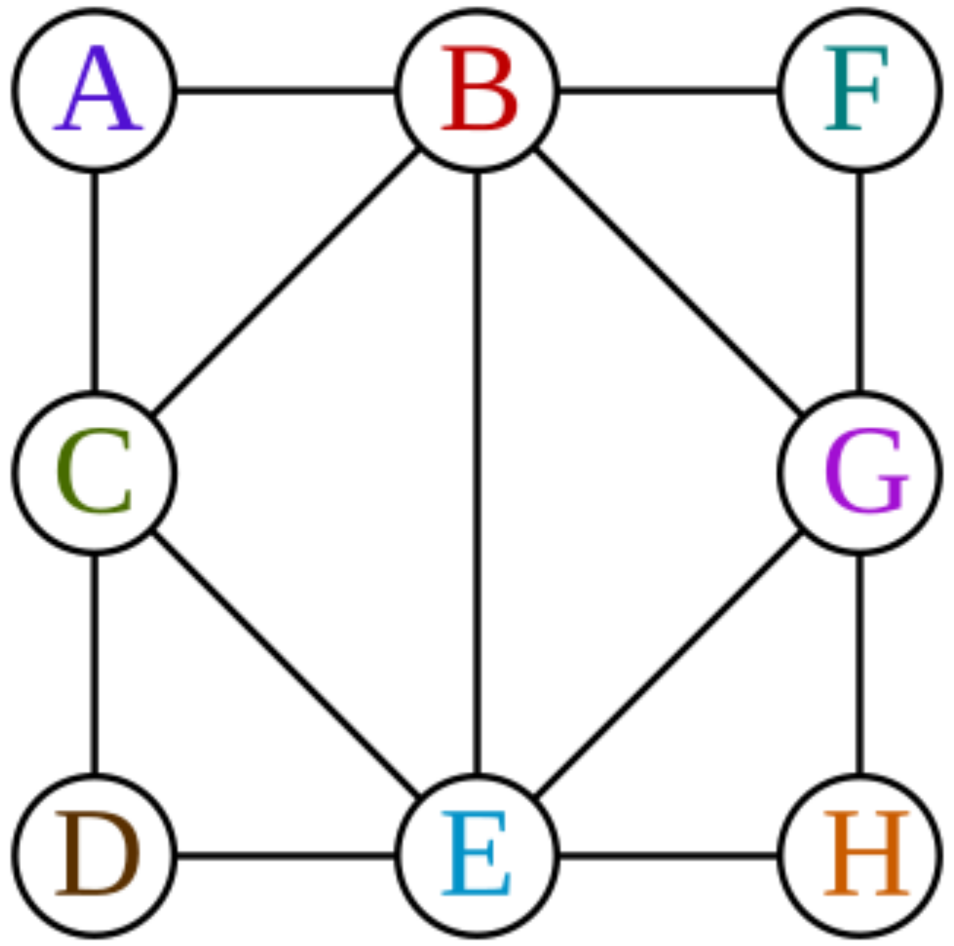
Applications

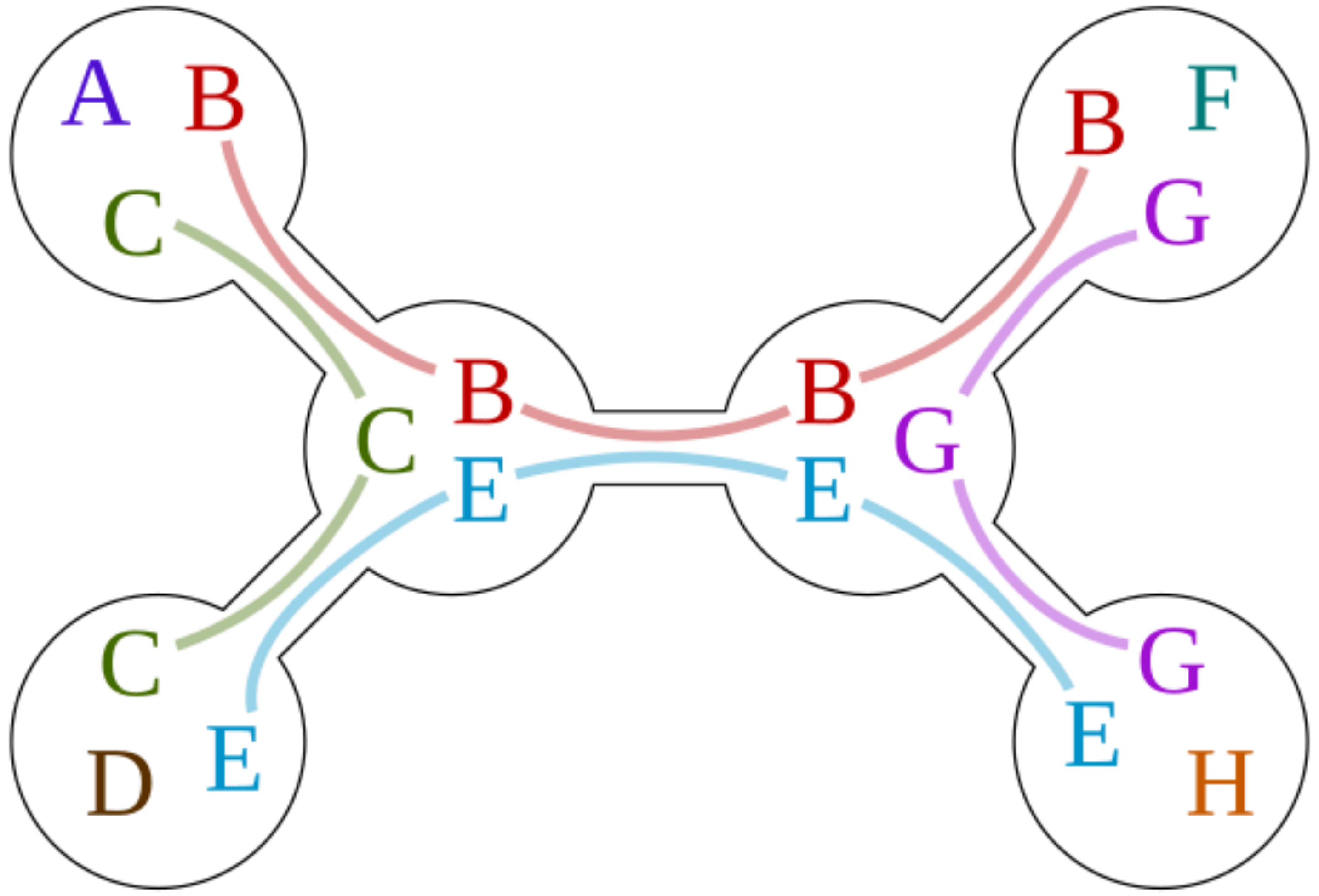
Problem	General	Bounded treewidth
Maximum IS	$\Omega(n^2/\log^2 n)$	$\tilde{\mathcal{O}}(D)$
Maximum Matching	$\tilde{\mathcal{O}}(\mu(G)^2)$	$\tilde{\mathcal{O}}(D)$
Minimum Vertex Cover	$\mathcal{O}(n^2/\log n)$	$\tilde{\mathcal{O}}(D)$

… Minimum Dominating Set, Maximum Clique, etc…

High Level Idea

TREE DECOMPOSITION: Pair (X,T). T is a tree, V(T)=X and $X \subseteq \mathscr{P}(V)$





(1) Union of bags equals V

(2) Every edge is contained in a bag

(3) For every node v, graph induced by bags containing v is connected

Courcelle's idea

Theorem. Let G be an n-node graph with $tw(G) \leq k$. Then G admits a tree decomposition (X, T) such that

- T has depth $O(\log n)$ and,
- For every node $t \in X$, the bag B_t has size $O(k)$.

ALGORITHM

(I) Compute a tree decomposition (X,T)

(II) Apply dynamic programming over the tree decomposition

Distributed implementation

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How to represent a tree decomposition?

How to build a tree decomposition?

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How to represent a tree decomposition?

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How to apply dynamic programming

In $\tilde{\mathcal{O}}(D)$ rounds?

Computing a tree decomposition

Definition. Given an n-node graph G a separator set S of G is a subset of nodes such that the size of each connected component of $G - S$ is at most $2n/3$

One Separator

Lemma (J. Li, P. Montealegre, I. Todinca, B.J.) There exists a CONGEST algorithm such that given a graph G with $tw(G) = k$ and any connected subgraph H, computes in $\tilde{\mathcal{O}}(D)$ rounds, a separator set of size at most k

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Disjoint Separators

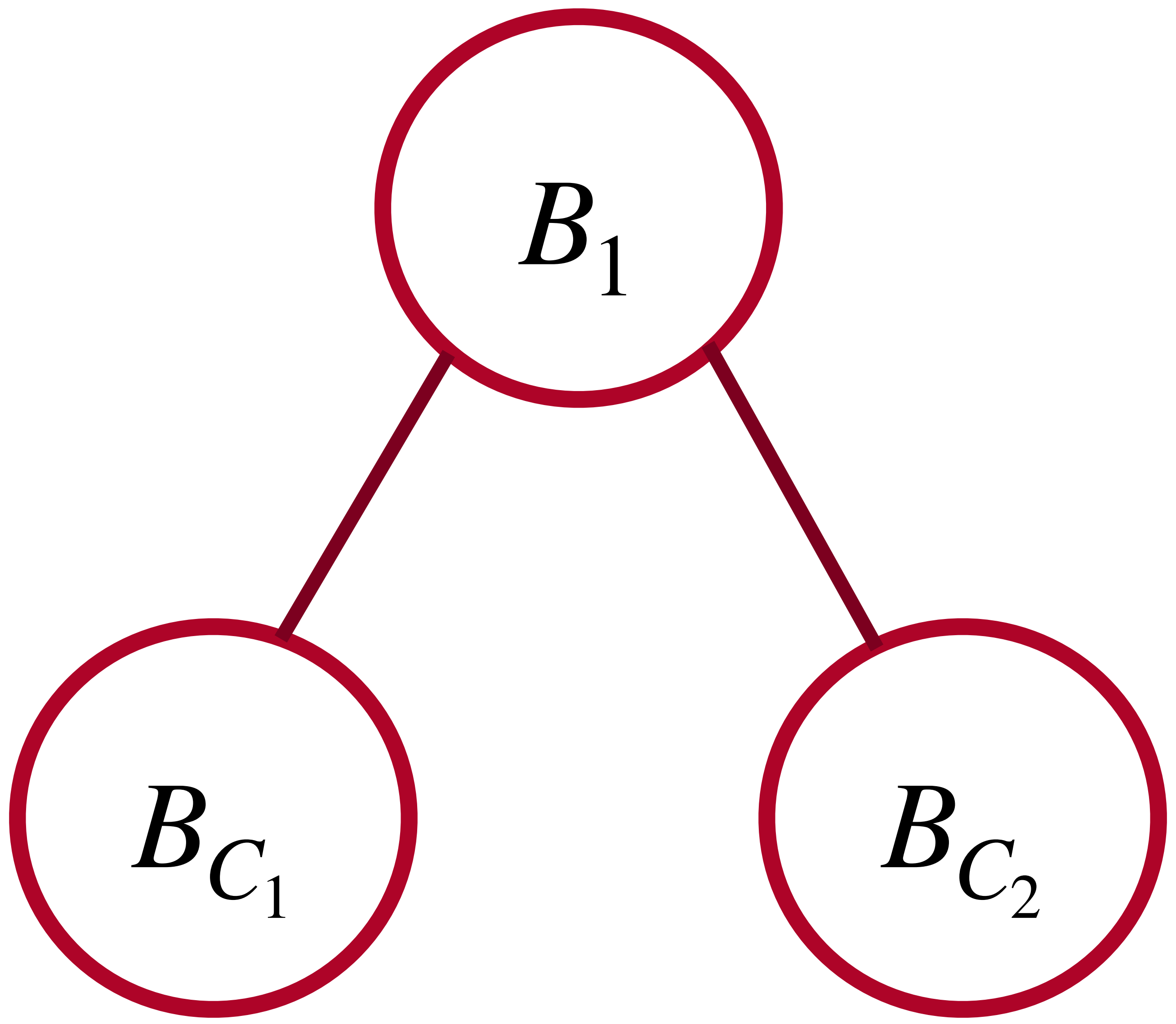
Lemma (J. Li, P. Montealegre, I. Todinca, B.J.) There exists a CONGEST algorithm such that given a graph G with $tw(G) = k$ and any collection $H_1, \ldots, H_j$ of disjoint and connected subgraphs, computes in $\tilde{\mathcal{O}}(D)$ rounds, in parallel, a separator set of size at most k

Distributed computation

Recurse in instances (U, X) with $U \subseteq V, X = N(U)$.

Start with $(V, \emptyset)$

- $S_V = separator(G)$
- $B_1 = S_V$ **(root bag)**
- **In parallel in each c.c.** C_i **of** $G - B_1$
  - $S_{C_i} = separator(C_i)$
  - $B_{C_i} = S_{C_i} \cup N\left(C_i\right)$ **(**)**
  - **Continue recursively in c.c. of** $C_i - S_{C_i}$

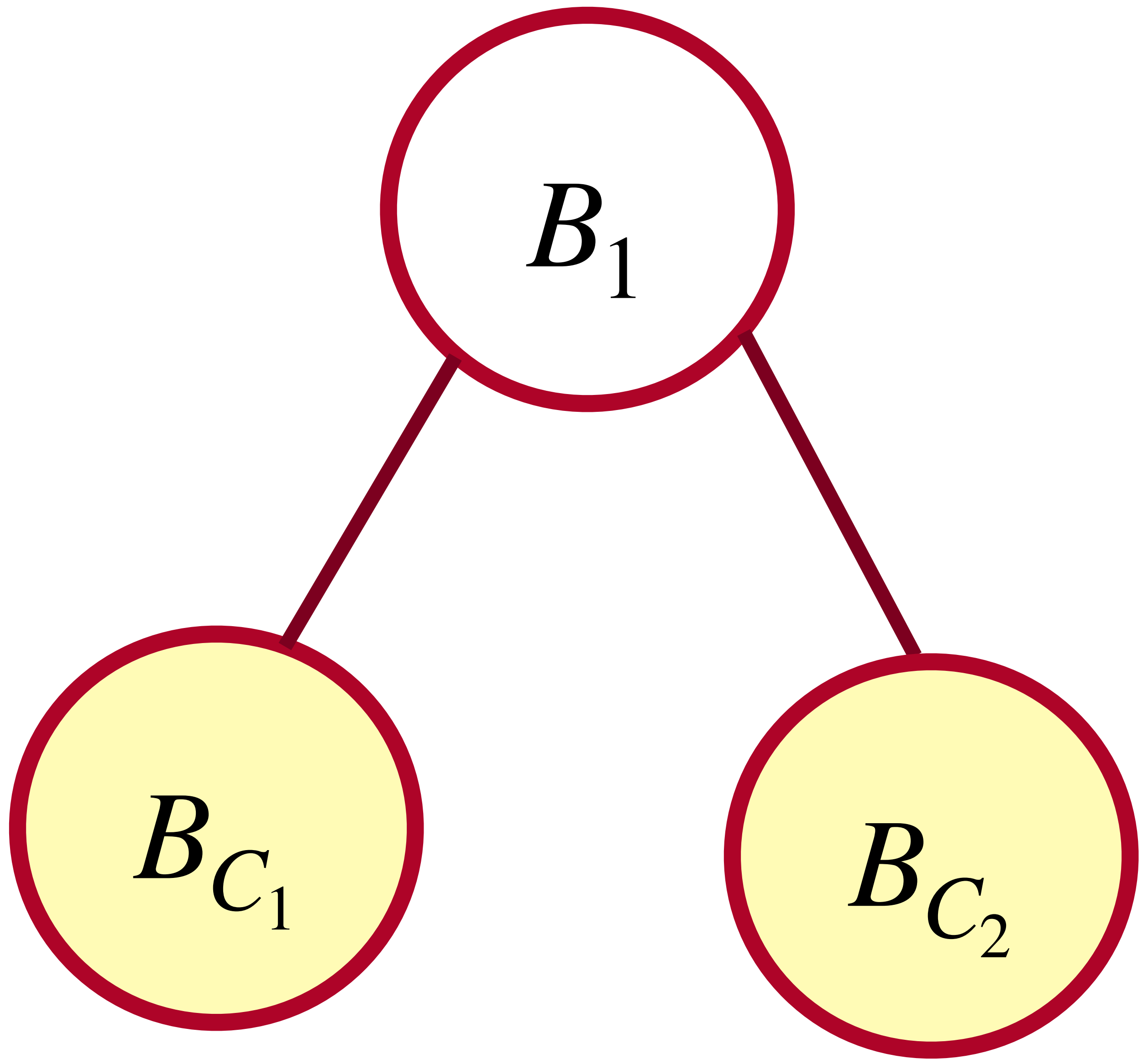


Properties

- **Size $\mathcal{O}(k)$ of each bag**
- **Depth $\mathcal{O}(\log n)$**
- **Inherently we obtain a distributed representation in each node**
- **Each bag has a leader**

Dynamic programming: Maximum Independet Set

(1) **Leaf bags compute all possible solutions greedly.**



Dynamic programming: Maximum Independet Set

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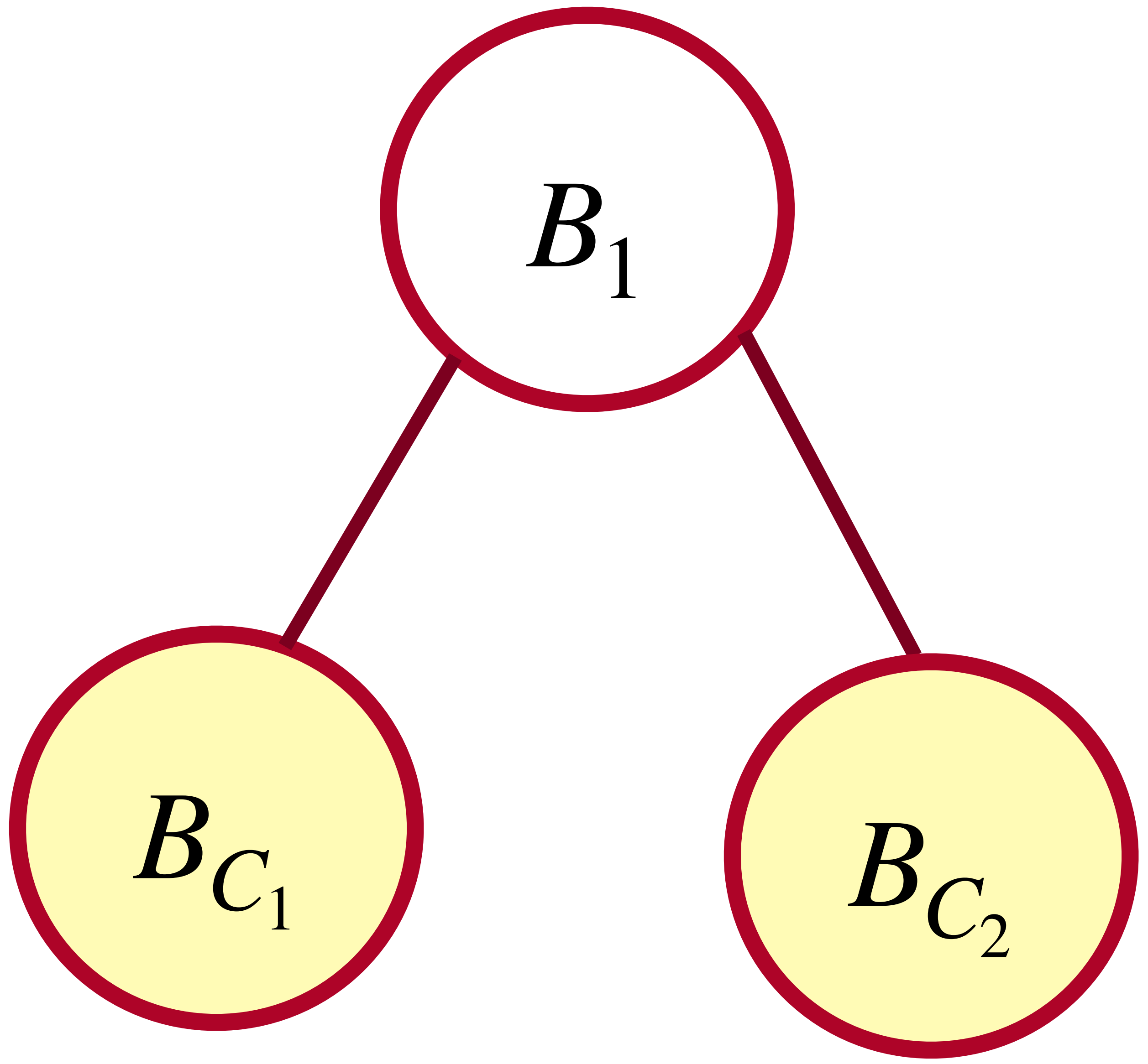
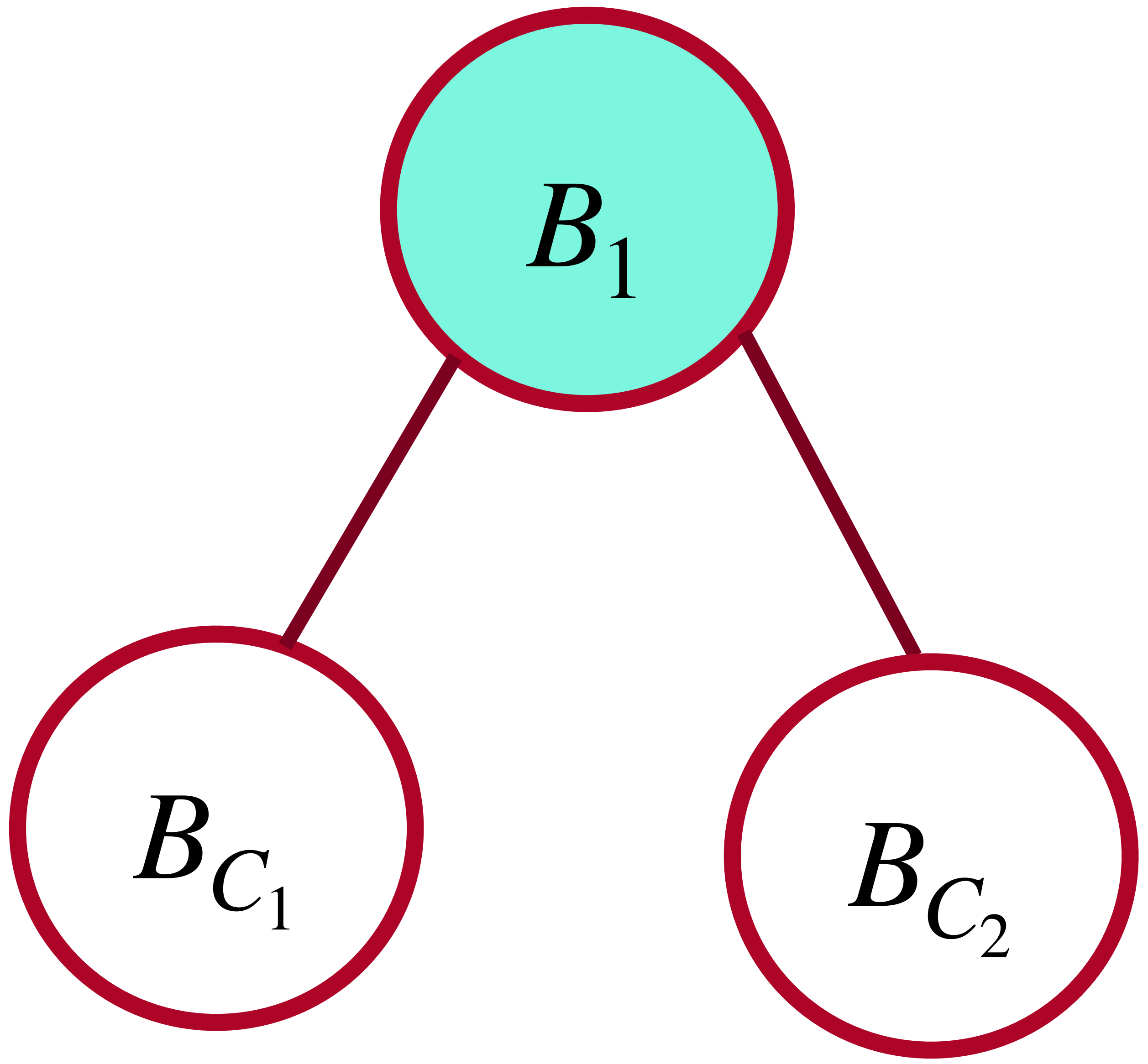
$$OPT(X_{C_i}) = \left| MaxIS\left(G\left[X_{C_i}\right]\right) \right|$$



$\boxed{2^{O(k)} \text{ rounds}}$

Dynamic programming: Maximum Independet Set

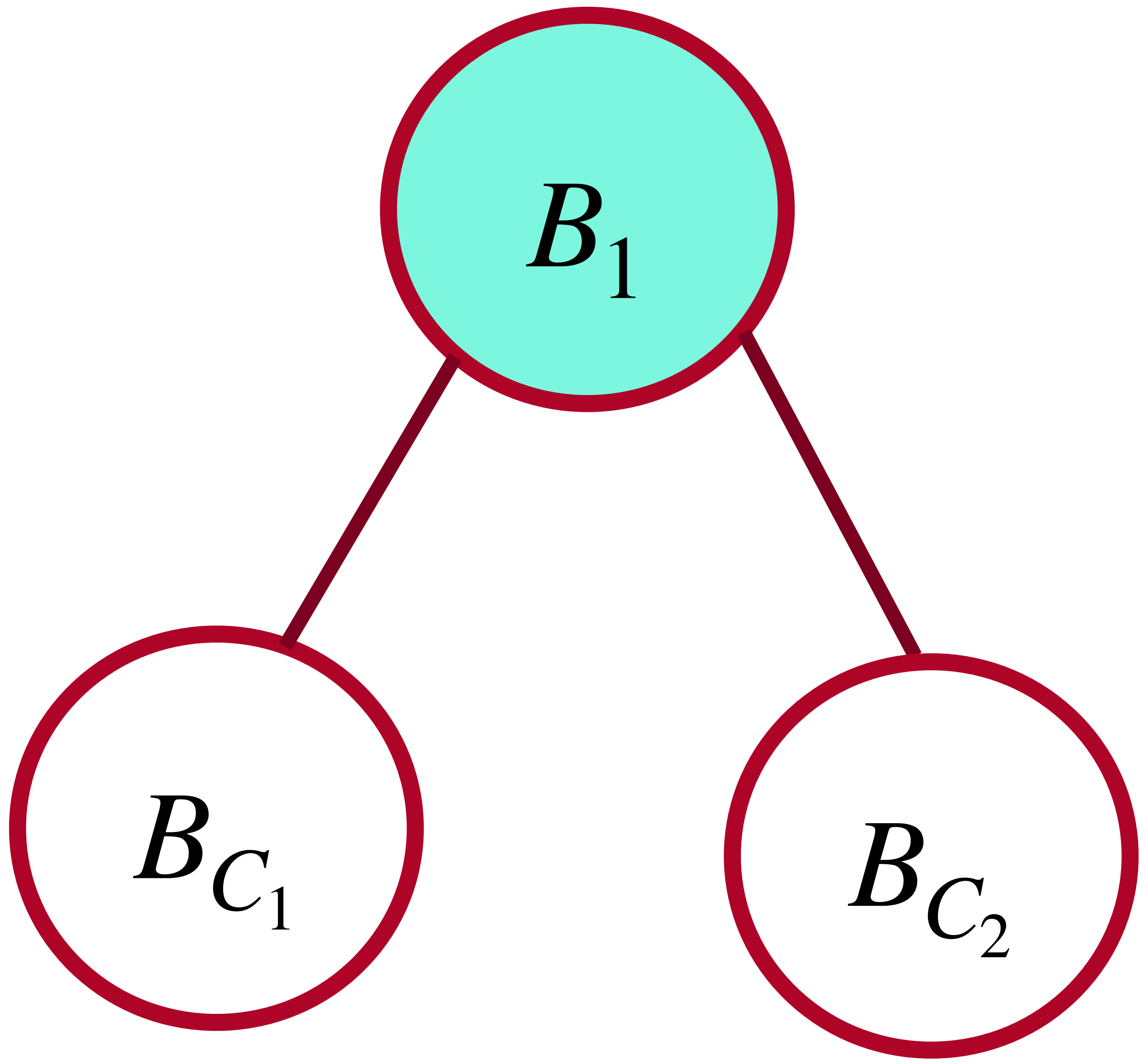
(2) Non-leaf bag: Extend the solution of its children



Dynamic programming: Maximum Indepedent Set

(2) Non-leaf bag: Extend the solution of its children

"Compute commutative and associative function of local input"



Great tool: Partwise aggregation problems

$\boxed{\tilde{O}(k \cdot D) \text{ rounds}}$

Summary

➡ Deterministic implementation of Courcelle's Theorem in CONGEST.

➡Improve to $\tilde{\mathcal{O}}(D)$ rounds several problems **with the same algorithm**

➡ Decision, optimization and **counting** version implemented.


MISAOTRA
KIITOS
MAHALO
THANK YOU
CHOKRANE
MATONDO
AŠANTE
DANKE
MÈSI
VINAKA
TAKK
MERCI
OBRIGADO
ARIGATO
DZIĘKUJĘ
MULTUMESC
ŠALAMAT
HVALA
TAK
ΕΥΧΑΡΙΣΤΩ
GRÀCIES
KÖSZÖNÖM
BEDANKT
TEŞEKKÜRLER
PALDIES
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Paper